Supplementary material on:

DOMINO EFFECT MOTION INVESTIGATION: A NUMERICAL APPROACH

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This document includes more information about the theory and modeling of the problem which is not appeared on the original paper.

Physical modeling:

Consider the forces applied to one domino (Figure 1):

These forces are:

1) Normal force *i-1* and *i* *2)* Friction force *i-1* and *i* *3)* Normal force *i* and *i+1* *4)* Friction force *i* and *i+1* *5)* Weight *6)* Normal force from surface *7)* Friction force between domino and the surface

The equation for the torque applied to the i^th domino is:

$$I\ddot{\theta}(i) = \mathrm{Torque}\big(\mathrm{F(i-1)},\ \mu F(i-1),\ F(i),\ \mu F(i), \mathrm{weight}\big)$$

More precisely:

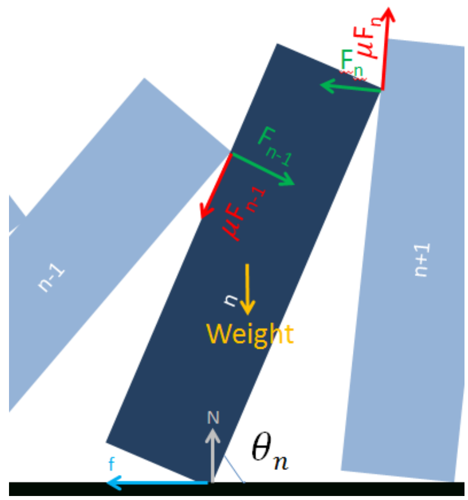


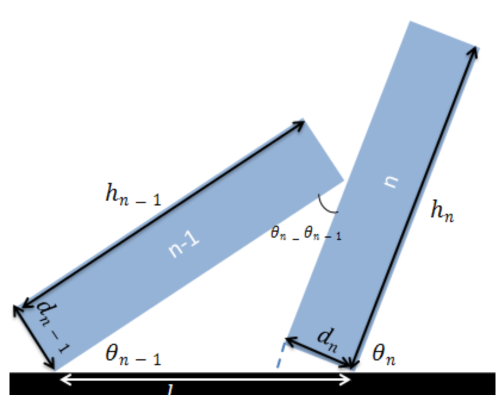
Figure 1: Forces applied to one domino

$$I(i)\ddot{\theta}(i) = -F(i-1)\times\left\{\left[\left(l(i-1)-\frac{d(i)}{\sin(\theta(i))}\right)^2 + \big(h(i-1)\big)^2 - 2h(i-1)\left(l(i-1)-\frac{d(i)}{\sin(\theta(i))}\right)\cos\big(\theta(i-1)\big)\right]^{0.5} - d(i)\cot\big(\theta(i)\big)\right\} + \mu F(i-1)d(i) + F(n)h(i)cos\big(\theta(i+1)-\theta(i)\big) + \mu F(n)h(i)sin\big(\theta(i+1)-\theta(i)\big) - m(i)g\frac{\sqrt{h(i)^2+d(i)^2}}{2}\cos(\theta(i)+\mathrm{atan}(\frac{d(i)}{h(i)})) \qquad (1)$$

The other equation is the geometric constraint that is calculated by sinus law:

$$\frac{h(i-1)}{sin\theta(i)} = \frac{l(i-1)-\dfrac{d(i)}{sin\theta(i)}}{\sin(\theta(i)-\theta(i-1))}$$

Second derivative of this equation against time results in:



$$\ddot{\theta}(i-1)+\ddot{\theta}(i)[-1+\frac{l(i-1)\cos(\theta(i))}{h(i-1)\sin(\theta(i)-\theta(i-1))}]=\frac{l(i-1)\dot{\theta^2}(i)}{h(i-1)\cos(\theta(i)-\theta(i-1))^2}[\sin(\theta(i))\cos(\theta(i)-\theta(i-1))^2-\frac{\cos(\theta(i))^2\sin(\theta(i)-\theta(i-1))}{h(i-1)}] \tag{2}$$

In better words, we can write two equations for each of dominoes.

$$C1(i)F(i-1)+C2(i)F(i)=C3(i) \tag{3}$$

$$P1(i)\ddot{\theta}(i-1)+P2(i)\ddot{\theta}(i)=P3(i) \tag{4}$$

Where C1(i), C2(i), C3(i), P1(i), P2(i), P3(i) are known constants.

The program has to solve this system of 2n equations (2 equations for each of dominoes) and 2n unknowns ($F(i)$, $\ddot{\theta}(i)$) in each time iteration. It is obvious that F(n)=0, F(0)=0.

Therefore if we make these matrices (examples of 4 dominoes are shown):

M =

0	0	C2(1)	0	0	0	0	0
0	P1(1)	0	P2(1)	0	0	0	0
0	0	C1(2)	0	C2(2)	0	0	0
0	0	0	P1(2)	0	P2(2)	0	0
0	0	0	0	C1(3)	0	C2(3)	0
0	0	0	0	0	P1(3)	0	P2(3)
0	0	0	0	0	0	C1(4)	0
0	0	0	0	0	0	0	P1(4)

$$X=\begin{matrix}F(1)\\ \ddot{\theta}(1)\\ F(2)\\ \ddot{\theta}(2)\\ F(3)\\ \ddot{\theta}(3)\\ F(4)\\ \ddot{\theta}(4)\end{matrix} \qquad B=C3(1)\quad P3(1)\quad C3(2)\quad P3(2)\quad C3(3)\quad P3(3)\quad C3(4)\quad P3(4)$$

This equation is available:

M=X*B

So X=B*inverse (M) and F(i) and $\ddot{\theta}(i)$ is found for all i<=n (i.e solving a 2n*2n matrix)

After solving this matrix in each of iterations, the program updates amounts of $\theta(i), \dot{\theta}(i)$ for all i<=n by using the following equations:

- $\dot{\theta}(i) = \dot{\theta}(i) + \ddot{\theta}(i) * dt$ (5)
- $\theta(i) = \theta(i) + \dot{\theta}(i) * dt$ (6)

Collision:

Several methods for modeling the collision were assumed and tested. The best method is presented. The equations used are Reservation of momentum in x direction and using the restitution coefficient. Reservation of momentum in x direction is available since the collision time is pretty small. We have:

$$\sum_1^n m(i)v(i) = \sum_1^n m(i)v'(i) + m(n+1)u \quad (7)$$

Which could be rephrased:

$$\sum_1^n m(i)h(i)|\dot{\theta}(i)|\sin(\theta(i)) = \sum_1^n m(i)h(i)|\dot{\theta}'(i)|\sin(\theta(i)) + m(n+1)h(n+1)|\dot{\theta}(n+1)| \quad (8)$$

First derivative of the geometric constraint against time gives relation between angular velocities of any 2 neighboring dominoes:

$$\dot{\theta}(i-1) = \dot{\theta}(i)[1 - \frac{l(i-1)\cos(\theta(i))}{h(i-1)\cos(\theta(i)-\theta(i-1)}] \quad (9)$$

This equation indicates that if $\dot{\theta}(i)$ decreases by coefficient α after the collision, $\dot{\theta}(i-1)$ will also decrease by the same amount. Therefore, if $\dot{\theta}'(n) = \alpha\dot{\theta}(n)$, it could be inferred that $\dot{\theta}'(i) = \alpha\dot{\theta}(i)$ for 0<i<=n

Re-writing equation (7) we get:

$$\sum_1^n m(i)h(i)|\dot{\theta}(i)|\sin(\theta(i)) = \alpha\sum_1^n m(i)h(i)|\dot{\theta}(i)|\sin(\theta(i)) + m(n+1)h(n+1)|\dot{\theta}(n+1)| \quad (10)$$

For simplifying we call S=$\sum_1^n m(i)h(i)|\dot{\theta}(i)|\sin(\theta(i))$. Equation (8) gives

$$S(1-\alpha) = m(n+1)h(n+1)|\dot{\theta}(n+1)| \quad (11)$$

The other equation is restitution coefficient. The relative velocities of n and (n+1) get –e times (e<1) after the collision.

$$-e\left(h(n)\left|\dot{\theta}(n)\right|\sin\left(\theta(n)\right) - 0\right) = h(n)\left|\dot{\theta}(n)\right|\sin\left(\theta(n)\right) - h(n+1)\dot{\theta}(n+1) \quad (12)$$

The coefficient e, is calibrated in experiments. Solving equations (11) and (12), with two unknowns of **α** and $\boldsymbol{\dot{\theta}(n+1)}$ two important parameters are calculated. The program updates $\dot{\theta}(n+1)$ from 0 to its new amount and also multiplies $\dot{\theta}(i)$ by the coefficient α for 0<i<=n.