Brilliant Pattern

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Abstract

This paper shows our investigation of an IYPT 2010 problem “Brilliant Pattern”. We show our results from experimental investigation - photos of observed light patterns, and try to explain their uncommon shape. Later we introduce a Catastrophe Theory which turns out to be the best explanation of observed phenomenon. We show some basic facts on this theory, examples and how important it was in investigation of caustics. Finally we classify seen patterns in terms of catastrophe theory.

Keywords

Patterns, light, catastrophe optics, catastrophe theory, caustics, geometrical optics, rainbow, water drop

Introduction

A drop of water is an interesting object and turns out to be a gateway to world of catastrophe optics [1]. One of the most common opportunities to look inside this world is right after rain, when sun comes out. In favorable atmospheric conditions a beautiful phenomenon can be observed – a rainbow. It is one of the basic structures investigated by catastrophe theory. As it turns out, there are many other light patters that can be observed when a drop of water is illuminated with laser light, those patterns are called caustics.

These structures were investigated for several years. Not only do they appear in optics but also in many other domains of our life such as economy or biology. This is why there was such a high interest among scientists to find a mathematical model that could describe them in a simple way. A revolutionary theory was presented in the 1960’s by a French mathematician René Thom. He developed a mathematical model that showed that the majority of common caustics can be transformed using smooth geometric transformations into one of seven elementary catastrophes.

A close study of light patterns created by laser light illuminating a drop reveals something more powerful: Catastrophes seen through a drop of water are actually cross-sections taken through a three dimensional structures, which in turn are cross-sections through mathematical objects of higher dimensions. In this article we will show at first our experimental results and than how observed light patterns are connected with the mentioned catastrophe theory.

Experimental Investigations

The main object of our study was a drop of water. It was created using a needle connected to a syringe with deionized water. Because laser light

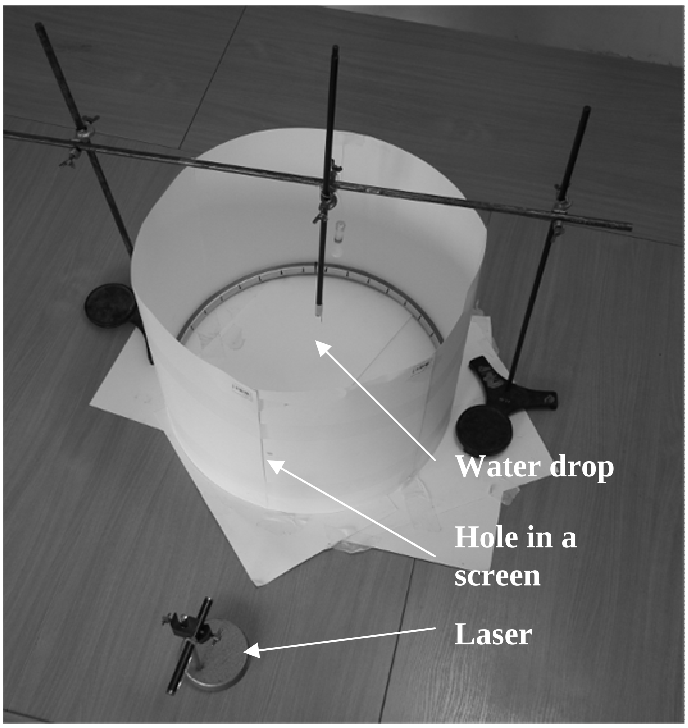

FIG 1: Experimental setup

pointed at a drop of water can be reflected and refracted in all directions, a cylindrical paper screen, 1m in diameter, was built around the drop. The base of the cylinder was also covered with paper, see FIG 1. The water drop was illuminated by green and red laser through a hole in a side of the screen. The whole experiment was conducted in a dark room; this helped us in observing relevant phenomena. Interesting light patterns were observed in this experiment. Photos of some of them are shown below (FIG 2).

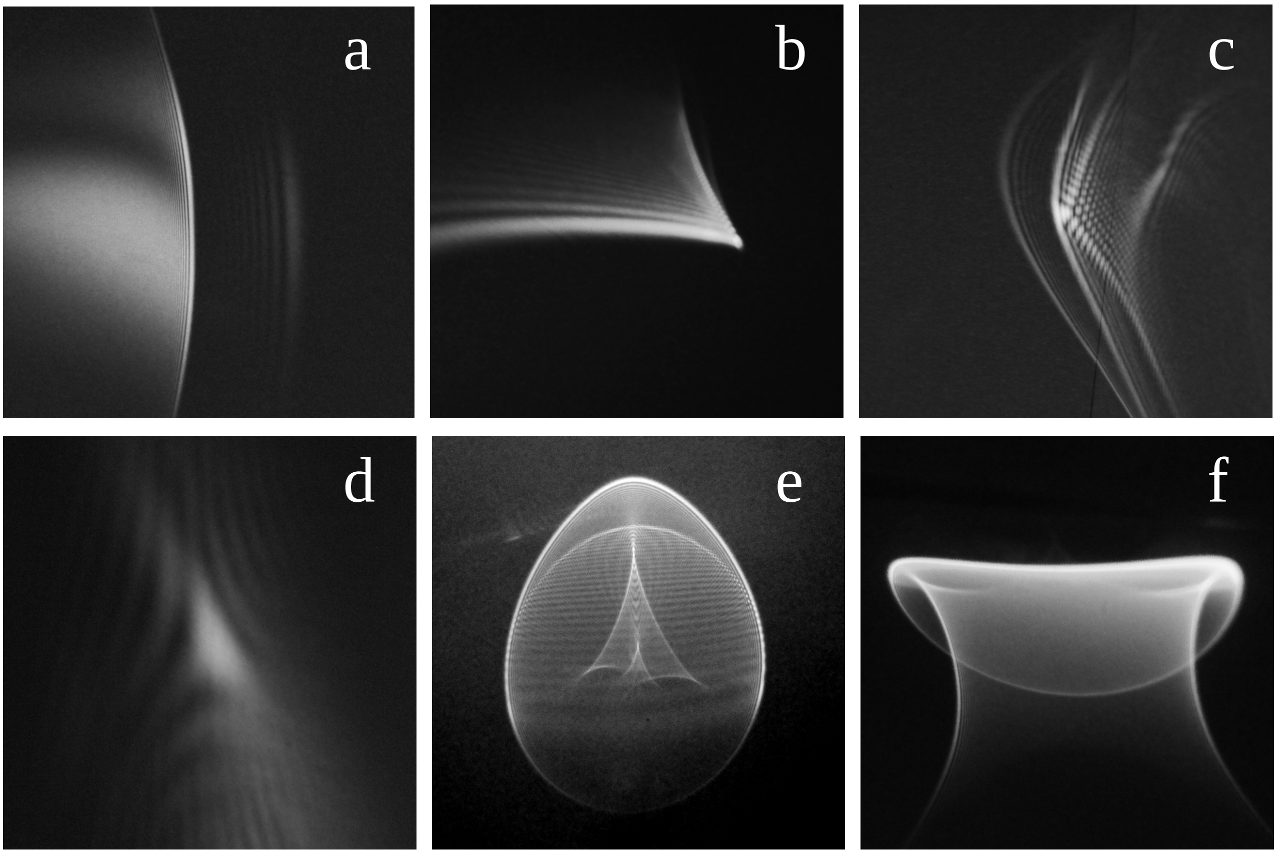

FIG 2 (a, b, c, d, e, f): Photos of patterns that were observed when a drop of water was illuminated with green laser light

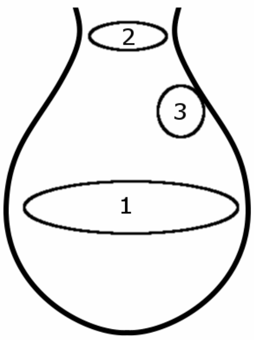

FIG 3: Drop of water divided into parts

The light patterns were observed in different conditions. Pattern 2a, was observed on the side of the screen when a drop of water was illuminated in part 1 (see FIG 3). Patterns 2b and 2c were observed on the side of the screen when part 2 of the drop was illuminated. Patterns 2e, 2f were observed on the bottom side of the screen when part 2 was illuminated. It is worth to mention that pattern 2e could be observed only when drop's size was being changed, never in static situation. Pattern 2f was seen in front of the drop on the side of the screen when part 3 of the drop was illuminated.

Seen structures appear to be complicated objects, and their variety seem to be endless, although some simple observations can be made. All of these light patterns consist of very bright curves, called caustics, which divide brighter parts of the screen from much darker parts. It can be easily seen in FIG 2a, since this is the simplest pattern. It seems obvious, that the interferential phenomena do occur in the discussed problem, but due to its complicity, we shall focus only on the geometrical aspects of the problem, using geometrical optics. As it will be shown later this approach gives satisfying results.

Theoretical Investigations – Catastrophe Theory

Catastrophes in science are sudden, qualitative changes of systems behavior with a smooth change in external conditions [2] (for example water begins to boil with a smooth change of its temperature). As it turns out, such phenomena are observed in optics (see examples in [3]). They can be analyzed by using Fermat's principle which says that the optical path length must be stationary: it can be either minimal, maximal or a point of inflection (a saddle point). Investigation of light paths (which can be found in [4]) shows that patterns created due to reflections and refractions of light are caustics which can be described as catastrophes. Catastrophe theory shows that all caustic of number of parameters less than or equal to 4 can be transformed using smooth geometric transformations to one of elementary catastrophes. Potential functions of elementary catastrophes are shown bellow [4]:

1. Fold $A_2(x) = \frac{1}{3}x^3 + ax$
2. Cusp $A_3(x) = \frac{1}{4}x^4 + \frac{1}{2}ax^2 + bx$
3. Swallowtail $A_4(x) = \frac{1}{5}x^5 + \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx$
4. Hyperbolic umbilic $D_4^-(x, y) = x^3 + y^3 + axy + bx + cy$
5. Elliptic umbilic $D_4^+(x, y) = x^3 - 3xy^2 + a(x^2 + y^2) + bx + cy$
6. Butterfly $A_5(x) = \frac{1}{6}x^6 + \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx$
7. Parabolic umbilic $D_5(x, y) = x^2y + y^4 + ax^2 + by^2 + cx + dy$

Close study of these caustics reveals something even more powerful. There are geometrical structures associated with elementary catastrophes, which describe dependence of number of extremes of these polynomials from parameters (a, b, c, d) in parametric space. Shapes created by taking a cross-section through these structures are similar to observed light patterns. In this situation parameters (a, b, c, d) are parameters connected with place in space where intensity of light is searched. Below an example of how to find those structures is shown.

Example: Cusp catastrophe

Cusp catastrophe potential function is a polynomial:

$$A_3(x) = \frac{1}{4}x^4 + \frac{1}{2}ax^2 + bx$$

We are interested in its critical points. In order to find them we have to solve an equation:

$$\frac{dA_3(x)}{dx} = 0$$

We find that:

$$x^3 + ax + b = 0$$

Plot of solutions to this equation was shown in space (x, a, b) in FIG 4. In order to find how number of extremes of A3 depends on parameters (a, b) plot was projected onto plane (a, b), creating two areas divided by a curve. Equation describing this curve is:

$$4a^3 + 27b^2 = 0$$

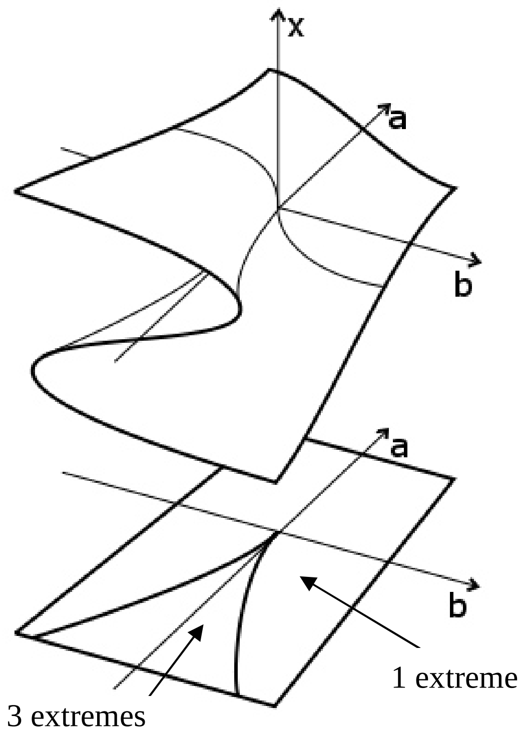

FIG 4: Cusp catastrophe extremes

This procedure can be repeated for other catastrophes. By projecting them onto parametric space (a, b, c) we find appropriate geometric structures. Below five of these structures are shown (except A5 and D5 where one more dimension “d” is needed).

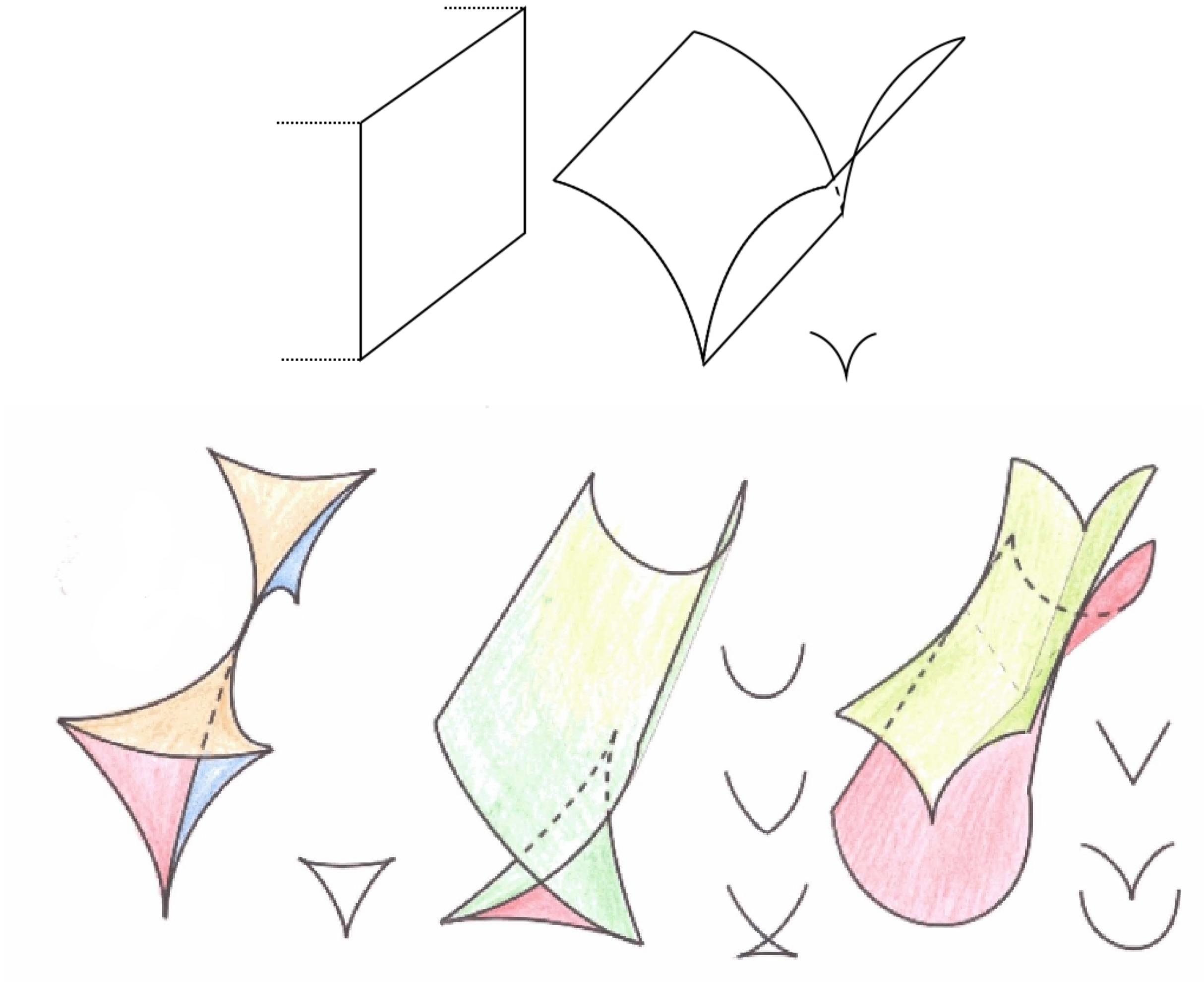

FIG 5: Structures associated with fold, cusps, elliptic umbilic, swallowtail, hyperbolic umbilic

Comparison between Theory and Experimental Results

Taking cross-section through those structures, different shapes can be observed. Some of them are shown in the FIG 5. Comparing those shapes to those seen in photos, some similarities can be found.

As it was mentioned, intensity of light depends on the number of rays that pass the point. Investigating the pattern in FIG 2a we observe that pattern seen on the screen can be divided in two areas: bright area where many rays are focused, dark area where no rays are focused. This is characteristic for fold catastrophe. For values of parameter $a<0$ there are 2 extremes of function A2, and for $a>0$ there are none. Deep analysis shows that pattern seen on this photo is a fold catastrophe. This pattern is also known as monochromatic rainbow. (First order and fifth order rainbow little to the right).

On the FIG 2b we can see a cusp catastrophe. This can be seen when comparing cross-section of a cusps catastrophe with this light pattern. What is interesting about this pattern is that it can be seen in every day life, when coffee-cup is illuminated with light.

FIG 2c shows a pattern which's nature is more complex than of the ones described above. Similar patterns can be achieved when taking a cross-section of a hyperbolic umbilic. In FIG 2d a pattern similar to a triangle star can be seen. It is an example of elliptic umbilic catastrophe.

Patterns seen in FIG 2e, 2f can not be characterized as one of the basic catastrophes, but it can be easily seen that those patterns are made of simpler forms that were described above.

Summary

Light patterns which are observed when a water drop is illuminated with laser light, have different and complex shapes. By finding similarities with catastrophe optics most of those patterns can be described in terms of this theory. Our investigation showed that a drop of water is a complicated object and it is able to create beautiful patterns. Although our model is not describing interferential effects it is a good base for such an examination.

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