Brilliant pattern

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ABSTRACT

Laser beam illuminating a drop of water is reflected and refracted on the air-water boundary. As a result various patterns are observed on the screen - build around the drop. Patterns were captured using a digital camera and phenomenon is described in terms of catastrophe theory. Basic facts about catastrophe theory are shown, and its results are compared with observed caustics.

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The author would like to dedicate this paper to memory of his teacher Mr. Stanisław Lipiński (the team leader of the Polish team to the IYPT 2010).

Introduction

Drop of water has been an interest of studies of many scientists, mostly because of its specific aerodynamic and optical properties. Problem of investigation of patterns that can be observed when suspended drop of water is illuminated with laser light was one of the problems for 23rd IYPT. Here the approach of the Polish team to this task is presented.

If one was to guess what patterns will be seen after the drop of water is illuminated with light beam, the first think to appear on his mind is the rainbow. This is a commonly known an optical phenomenon. Nevertheless it can be shown, that rainbow is only one of numerous optical patterns that can be observed during the study under laboratory conditions. Suspended drop of water has a complicated shape and it is hard to describe how it affects light paths. In some cases it can be approximated by a ball, but most of interesting patters are created due to drop's specific tear like shape. Because of complex structure of seen patters and complicated shape of the drop, it is hard to describe the phenomena in means of mathematical formulations. However there are several clues suggesting that observed patters are examples of optical catastrophes. Similar patterns created by a drop of water placed on the microscopic glass and illuminated from bottom were earlier investigated and were described as optical catastrophes [1, 2].

In this article we present an experimental setup used to recover light patterns coming out of a water drop. Out of wide range of observed patterns, most common were chosen and discussed. Images were analyzed in terms of catastrophe theory, and compared with elementary catastrophes.

Experimental investigation

Used in study experimental setup is shown on figure 1. The drop of water was created by a needle attached to a laboratory stand and connected to a syringe with deionized water. Because the laser light pointed at a drop of water can be reflected and refracted in any direction, a cylindrical paper screen, 1 m in diameter, with a paper bottom was built around the drop. The water drop was illuminated by green laser beam through a small hole in a side of the screen. All experiments were conducted in a dark room. Light patterns appeared on the screen depended on the place of illumination and the shape of the drop. Drop's diameter was adjusted from 1 to 7 mm using a syringe, larger drops could not be hold on a needle and felled down. The different parts of the drop were illuminated, by changing laser's beam direction. Some patters appeared only in dynamic condition, i.e. when the drop's size was continuously changed. Observed patterns were captured and recorded with a digital camera. For some specific experiments, instead of the water drop a fine optical glass ball of 5 mm in diameter was used. This way significance of drop's shape on the phenomenon was determined.

Pictures of the most interesting and characteristic patterns are shown on figure 2. Figure 3 illustrates place where drop of water was illuminated and the figure 4 place on the screen where a given pattern appeared. All of these light patterns consist of very bright curves, called caustics, which divide brighter parts of the screen from much darker parts (see figure 2a). Pattern on figure 2a is a caustic being approximately a straight-vertical line. It appeared when the drop was illuminated at the middle of the height (figure 3). This pattern was the easiest one to observe, it appeared in placed on the screen marked as ***a***, see figure 4. Pattern presented on figure 2b was a caustic similar to a cusp. Again one can notice that brighter parts of the screen are separated from darker by a bright curve. Pattern 2c has more complex structure than ones presented above. It consists of curves whose shapes are similar to parabola or cusp. Pattern 2d has shape similar to a triangle with interferential pattern around it. This pattern was hard to capture as it was very sensible to the place of drop's illumination (see place **d** on figure 3); a slight vibration (created by traffic on the street) interrupted the image. Pattern 2e appeared only when drop's size was being changed, never in stationary conditions. Image on figure 2e consists of egg shaped caustic with additional patterns inside, forming a shape similar to an arrow. Pattern 2f had different shape for small and big drops. When drop was small it had a parabolic shape with its arms facet outward the drop. When drop was bigger it began to form shape seen on figure 2f. Photo seen of figure 2f was taken right before drop detached from a needle.

When drop of water was replaced with a fine glass optical ball, pattern seen on figure 2a was the only one caustic observed. This shows that the specific drop's shape is relevant to disused patterns.

Theoretical Description

Parallel light rays of same wavelength, directed on a spherical drop of water are reflected and refracted on water air boundary. It seems obvious, that the interferential phenomena do occur in the observed images, but we focused on the geometrical aspects of the phenomenon. Rays that are only once reflected inside the drop form a first order rainbow, which is the most familiar caustic of all. Applying law of reflection and Snell's law, formula describing final direction of each ray can be found [3]:

$$\theta = \pi + 2\alpha - 4\arcsin\left(\frac{\sin(\alpha)}{n}\right); \tag{1}$$

where: n – relative refractive index between water and air, α – angle of incidence of initial ray on a drop, θ – angle between incidence light ray and one coming out of a drop.

This function is plotted on figure 5 (for case of water drop in air). One can see that for values of θ larger than critical value ($\theta_{cr} \approx 137.5^{o}$) there two values of α satisfying the equation, for smaller values there are none. A bifurcation occurs at value of θ equal to critical angle - with increasing value of parameter θ two solutions merge into one and there is no solution of equation (1) when the $\theta < \theta_{cr}$. Rays that travel near the bifurcation form a caustic which appears on a screen as a bright line. The value of critical angle is different for different wave lengths; therefore if drop is illuminated with white light, each color forms a caustic at a slightly different angle. All those colors together form a colorful rainbow.

If multiple reflections inside the drop are considered, second and higher order rainbows can be described [3]. In conducted experiments we were able to observe several rainbows up to a very dim 6-th order rainbow. On figure 2a the first and fifth (on the dark, right side of first) order rainbow is seen.

The rainbow is the only pattern that can be described using spherical drop model. Description of patterns presented on figure 2b, c, d, e, f requires more advance theory. As it was shown in case of a rainbow, bright curves called caustics appear in places where many rays intersect the screen. In mathematical formulation those are critical points of functions ilustrating rays direction. Several researchers, notably M. V. Berry and J. F. Nye, investigated such light patterns and came to a conclusion that those patterns can be explained in means of catastrophe theory.

Catastrophe theory

Catastrophe is defined as a sudden, qualitative change of the system's behavior with a smooth change in external conditions [4] (for example water begins to boil with a smooth change of temperature). Catastrophe theory analyses so called degenerated critical points. For a one variable function those are values of an argument for which not only first but second or more of higher-order derivatives become zero. Those points correspond to formation of caustics.

Caustics form at bifurcations which divide areas where number of rays crossing each point differs by two. In order to determine number of rays passing through each point; all light rays are taken under consideration. Their optical path length depends on initial and end position of a light ray. Initial position of the ray (out of a parallel bunch of parallel rays) can be described by two coordinates (X, Y), and the end position (position on the screen) by three coordinates (p, q, r). On the basis of Fermat's principle, the path of light between two points is extreme (a minimum, maximum, or other type of critical point) with respect to time required to travel it. Using this principle it is possible to find the number of light rays, coming from (X, Y), reaching a given point in space determined by (p, q, r).

In most cases the function describing optical paths is complicated, and usually its expansion in Taylor series around interesting points - critical points - is investigated. Details of this procedure are presented in reference [5]. Catastrophe theory states that all such functions with 4 or less control parameters can be transformed around critical points, to a one of seven forms called elementary catastrophes, using only smooth transformations. Five elementary catastrophes of our interest are listed bellow:

1. Fold $A_2(x) = \frac{1}{3}x^3 + ax$
2. Cusp $A_3(x) = \frac{1}{4}x^4 + \frac{1}{2}ax^2 + bx$
3. Swallowtail $A_4(x) = \frac{1}{5}x^5 + \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx$
4. Hyperbolic umbilic $D_4^+(x, y) = x^3 + y^3 + axy + bx + cy$
5. Elliptic umbilic $D_4^-(x, y) = x^3 - 3xy^2 + a\left(x^2 + y^2\right) + bx + cy$

Variables x, y and parameters a, b, c are connected with coordinates X, Y and p, q, r by smooth transformations.

The number of extremes of catastrophes depends on value of parameters a, b, c. (In discussed case how many light rays cross given point of space.) In (a, b, c)-parameter space, boundaries between regions where the number of extremes differs by two - are surfaces, and define a bifurcation set of a given catastrophe. On figure 7 bifurcation sets associated with listed above elementary catastrophes are shown in (a, b, c)-parameter space. For example of elliptic umbilic, if a point (a, b, c) is inside “triangular pyramid” there are two critical points (two light rays), if point (a, b, c) lies outside there are four (four light rays).

Comparison of observed patterns with elementary catastrophes

A pictures presented in Figures 2 are the most characteristic representatives of images captured during experiments. In performed experiments used screen was a two dimensional object, so patterns observed on the screen were cross-sections of three dimensional structures. Explanation of obtained images in terms of catastrophe theory, they have to be compared with the cross-sections of the bifurcation sets of catastrophes. As one can see, cross sections of the bifurcation sets (see figure 7) and patterns seen on the figure 2 reveal several similarities. While discussing first-order rainbow (figure 2a) it was shown that caustic appears on the boundary between region where there are two rays and no ray leaving a drop. This is an example of the fold catastrophe where a cross section is a line. It is a boundary between values of parameter ***a*** where there are two extremes and where there are none. Pattern seen on figure 2b is an example of cup’s catastrophe. Its caustic is similar to a cross-section taken through a cups catastrophe (figure 7b). By analogy to fold catastrophe one may expect that observed caustic is boundary between regions where number of rays crossing the screen through each point differs by two (bright-dark side). However it is not possible to proof experimentally how many rays cross the screen. Analogously patter 2c was found to be example of hyperbolic umbilic, 2d was found to be example of elliptic umbilic. Patterns seen on figure 2e and 2f, could not be characterized as one of elementary catastrophe. They appear to be more complex structures, which are composed of simpler forms described above such as fold and cups catastrophes.

In mathematic formulation caustics correspond to critical points of functions describing light rays which contribute to observed patterns. Such critical points in optics were investigated, and several scientists came to conclusion that they are well explained by catastrophe theory. In investigated case function describing light paths can not be found easily due to complicated shape of a drop. However it seems that this function has several critical points that appear on the screen as caustics. Those light patterns are similar to structures obtained from catastrophe theory.

Conclusions

We constructed an experimental setup, and recover optical patterns created by laser beam scattered on a drop of water suspended on a needle. Several light patterns were captured. It was concludes that bright curves in observed patterns are caustics. The simples caustic appearing on the screen as a straight line pattern was labeled as rainbow and described in terms of spherical drop model. Observed patterns were analyzed in means of catastrophe theory. It can be concluded that observed light patterns are elementary catastrophes or composition of those elementary forms.

References

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Figure 1: Experimental setup

Figure 2: Pictures of patterns that were observed when a drop of water was illuminated with green laser light

Figure 3: Drawing presenting where laser beam was directed to create patterns a, b, c, d, e, and f seen on figure 2.

Figure 4: Drawing presenting places where patterns a, b, c, d, e, and f appeared on the screen. Patterns e and f appeared on the bottom of the screen

Figure 5: Dependency of θ angle vs. incidence angle α for n = 1.33.

Figure 6: Drawing illustrating trajectories of light rays illuminating spherical drop of water.

Figure 7: Bifurcation sets of elementary catastrophes: fold, cups, elliptic umbilic, swallowtail, hyperbolic umbilic. Exemplary cross-sections are shown next to each structure.

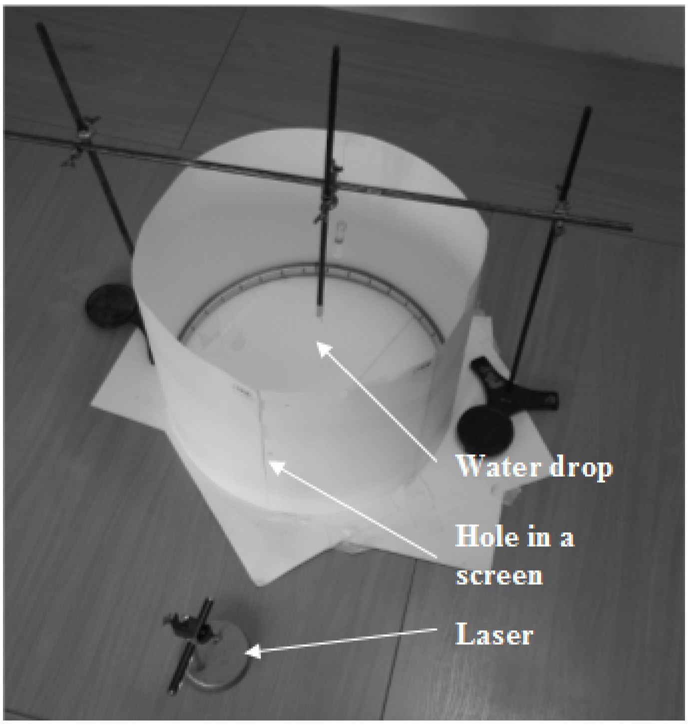

Figure 1: Experimental setup

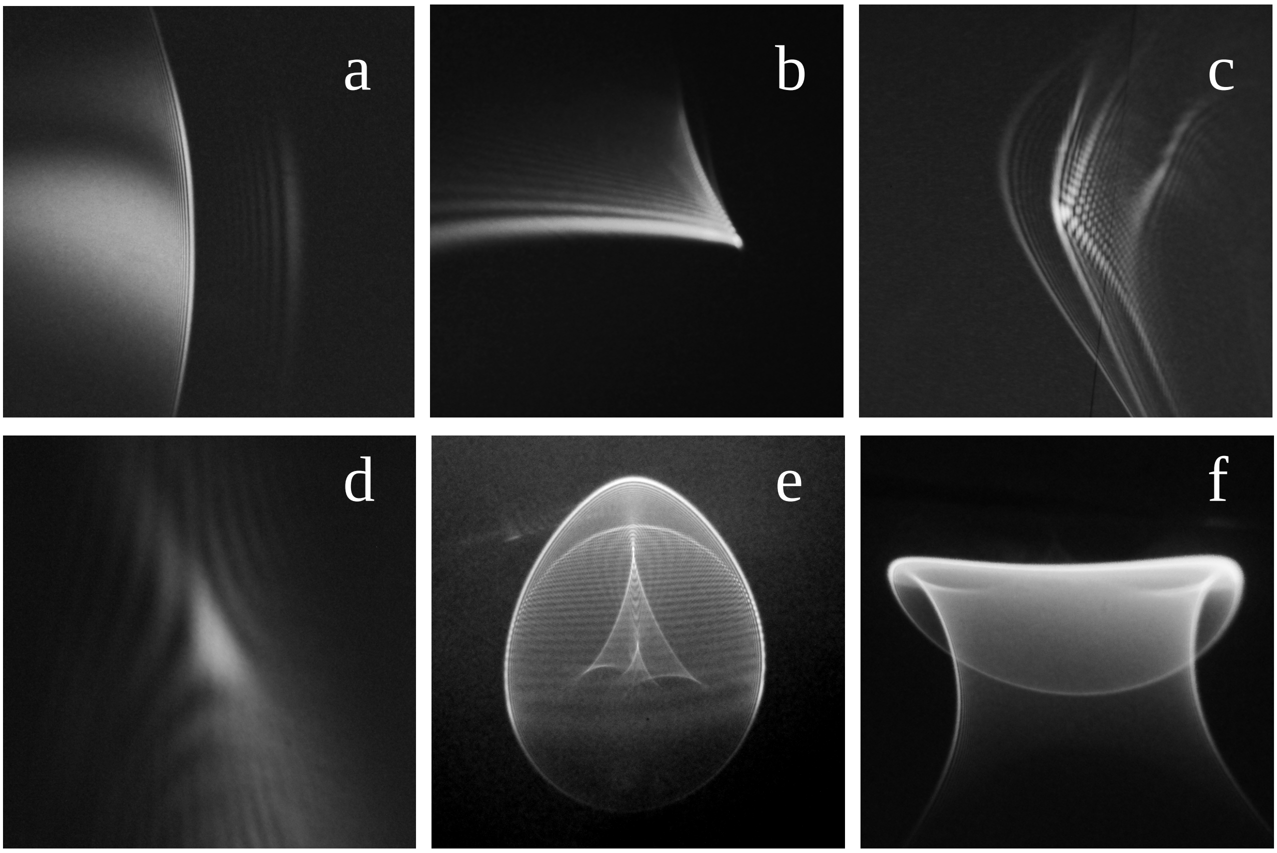

Figure 2: Pictures of patterns that were observed when a drop of water was illuminated with green laser light

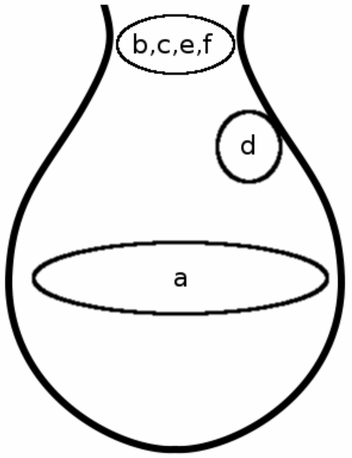

Figure 3: Drawing presenting where laser beam was directed to create patterns a, b, c, d, e, and f seen on figure 2.

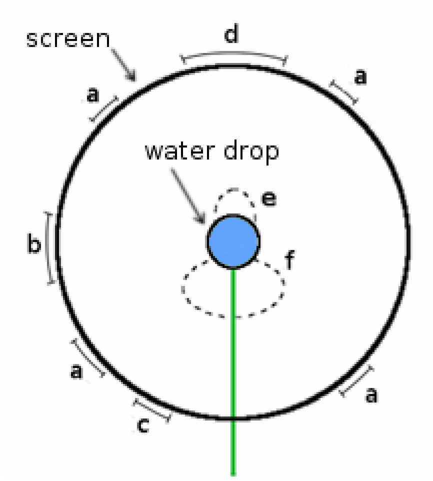

Figure 4: Drawing presenting places where patterns a, b, c, d, e, and f appeared on the screen. Patterns e and f appeared on the bottom of the screen

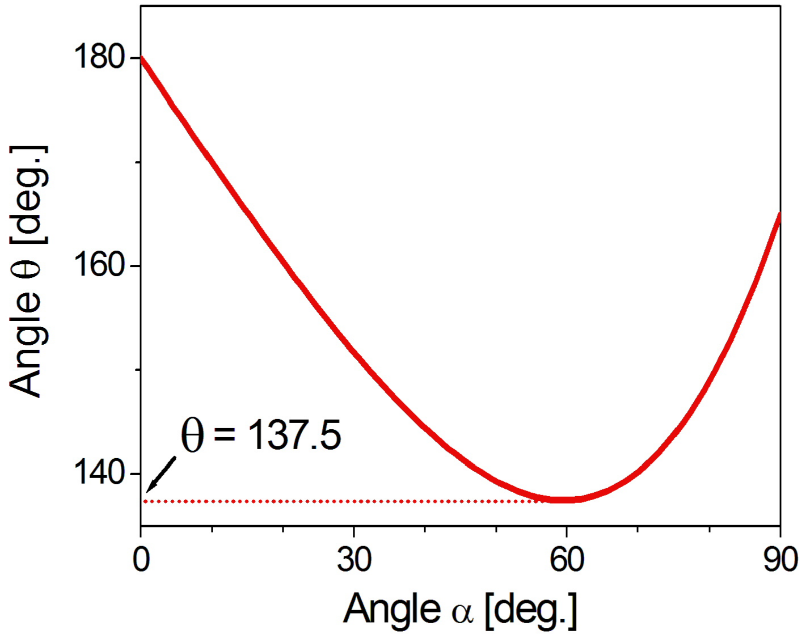

Figure 5: Dependency of θ angle vs. incidence angle α for n = 1.33.

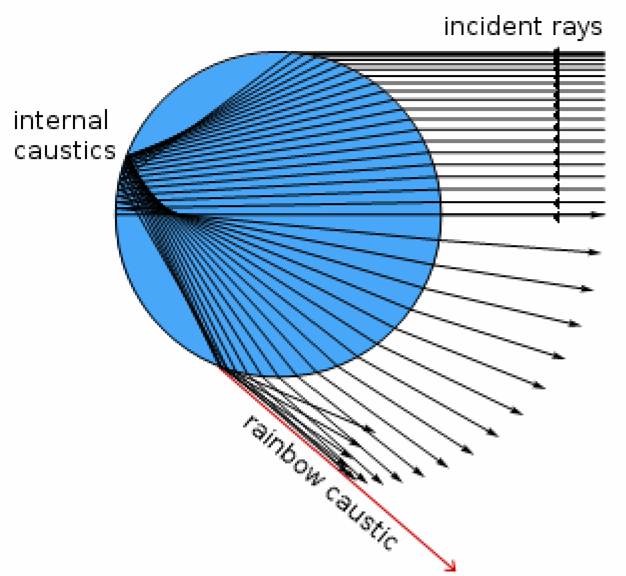

Figure 6: Drawing illustrating trajectories of light rays illuminating spherical drop of water.

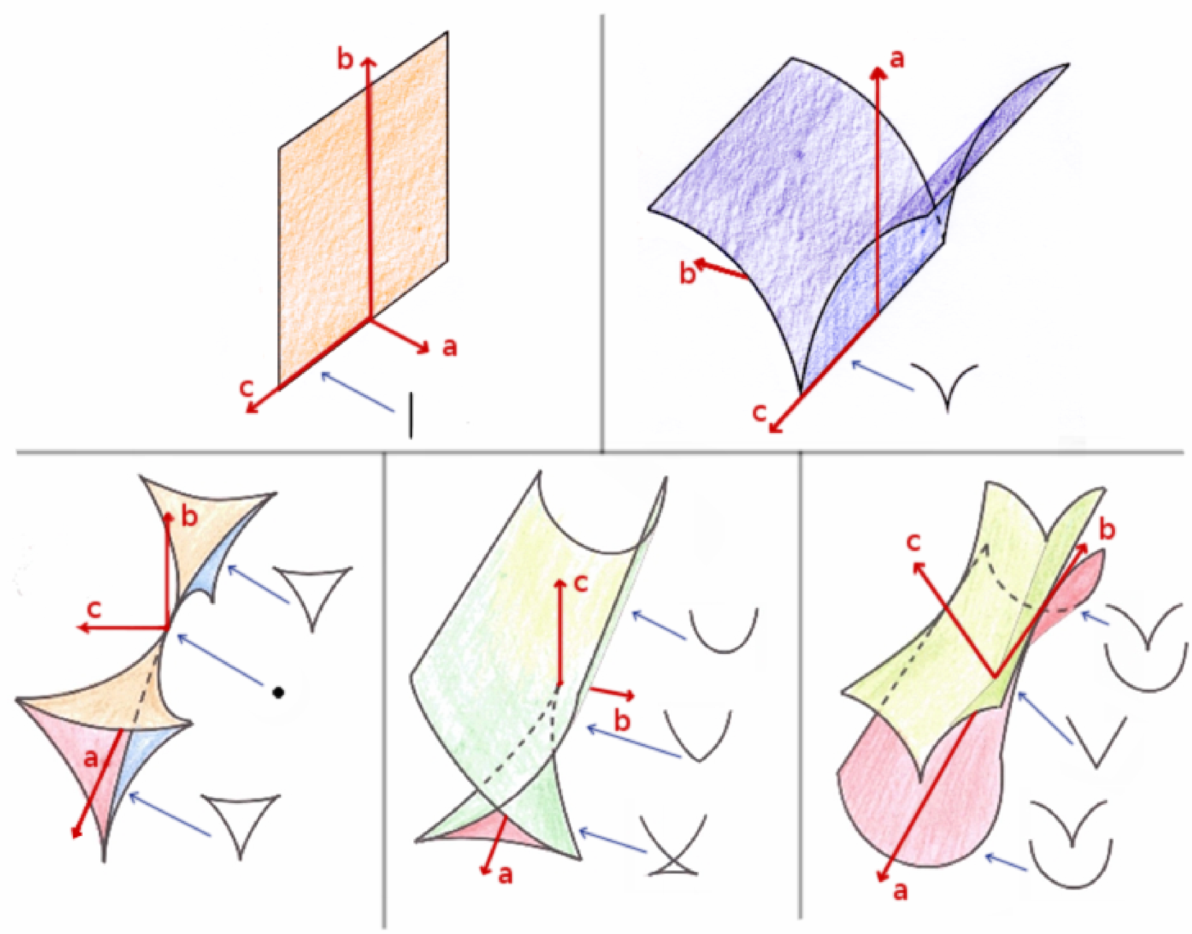

Figure 7: Bifurcation sets of elementary catastrophes: fold, cups, elliptic umbilic, swallowtail, hyperbolic umbilic. Exemplary cross-sections are shown next to each structure.