EXPANSION OF HORIZONTALLY ROTATING HELICAL SPRINGS

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Abstract
The present paper is a solution to IYPT 2010 problem no. 16, “Rotating Spring”. The main objective is to investigate the expansion of a helical spring rotated about one of its ends around a vertical axis. The effect of an additional mass attached to its free end is also a subject of investigation. This investigation is done in means of approximations leading to an analytical solution, as well as a developed numerical solution solving the differential equations assuming equilibrium of the forces in the rotating coordinate system. Both the analytical and numerical solutions are compared to physical experiments made by the author to examine the theoretical achievements.

Introduction

“A helical spring is rotated about one of its ends around a vertical axis. Investigate the expansion of the spring with and without an additional mass attached to its free end.”

Assuming the equilibrium condition in the rotating coordinate system, three forces would be exerted to every differential mass. The gravitational force, the spring force caused by its deformation, and the figurative centrifugal force that must be considered since an accelerated coordinate system is being used. The spring’s tensile force will be calculated assuming the Hooke’s law. This law is applicable in a limited range of strain among the spring; the range in which it remains elastic. So the spring’s tensile force will be a function of the modulus of the spring (μ), the spring’s initial length (l) and the change of length (Δl).

$$F_{Spring} = -\mu \frac{\Delta l}{l} \quad (1)$$

Note that the modulus was used instead of the spring’s constant because of being independent on the spring’s dimensions. For the same reason, the spring’s linear density will be used as the parameter instead of the spring’s mass.

The solution of this problem in the case where the mass of the spring causes extra tension and strain is complicated to solve analytically (if possible at all). To solve this general case, we will later bring a numerical solution. However, initially an approximation will be given to describe the cases when the mass of the spring is negligible compared to the additional mass attached. In this case, the spring could be considered as a whole, absolutely linear and weightless. So the three forces acting on the additional mass must be in equilibrium.

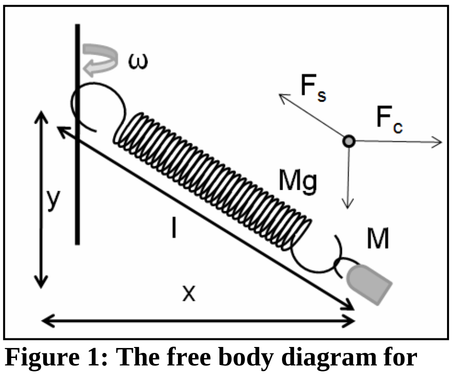

Figure 1: The free body diagram for the additional mass in equilibrium

Assuming that the sum of forces on the additional mass equal zero, in the case where $x = 0$, the length of the spring would be:

$$l_{\min} = l_0 + \frac{mg}{k} \quad (2)$$

And in the case where x does not equal 0,

$$l_{apx} = \frac{l_0 k}{k - m\omega^2} \quad (3)$$

Since the length cannot be less than the amount where x = 0:

$$l = \max(\frac{l_0 k}{k - m\omega^2}, l_0 + \frac{mg}{k}) \quad (4)$$

This result will be addressed as the analytical solution and will be compared to the physical experiments in the discussion.

Governing Equations

Because of the mass distribution of the spring along its length, the force needed to hold and rotate the free side of the spring differs in different points of its length. Thus the tension of the spring is not constant, making the linear density also a function of position. So the mass distribution of the spring its self is a function of the tension along the spring.

The equilibrium condition states that for each differential mass or length on the spring, the sum of the forces must be zero. This sum is the gravity and centrifugal force, plus the force caused by changes of tension along the differential length.

$$(\lambda \cdot x \cdot \omega^2)\hat{x} - (\lambda \cdot g)\hat{y} - \frac{d\hat{T}}{dl} = 0 \quad (5)$$

T is spring's tension, as a function of position. ω is the angular velocity. λ is the linear density, a function of position as well. *T* may be assumed to be in the same direction of the spring length, i.e. the spring does not stand any tension parallel to its length. Thus:

$$\frac{T_y}{T_x} = \frac{dy}{dx} \quad (6)$$

Derived from the Hooke's law, the linear density of the spring as a function of T would be:

$$\lambda = \frac{\mu \cdot \lambda_0}{\mu + T} \quad (7)$$

And the limitation of the boundary is the total mass of the spring:

$$\int_0^L \lambda(l)dl = \lambda_0 l_0 \quad (8)$$

L is the length of the spring after expansion, which is supposed to be the main unknown parameter to solve.

Numerical Solution

To solve the equation, the continuous medium of the spring is converted to a discreet medium. The spring is divided to *n* parts with the same initial length and mass. So the system is converted to n points which are connected with differentially small springs. Each point is being pulled by two springs. The position of equilibrium for each point is to be found. Transient method was used by to solve the problem; releasing the spring from an unstable position and iterating towards equilibrium. During iteration, every point will move towards the direction of the sum of the forces exerted to the point. The repetition of the iterations goes on until the movement of the points approach zero, i.e. the stable equilibrium condition is achieved.

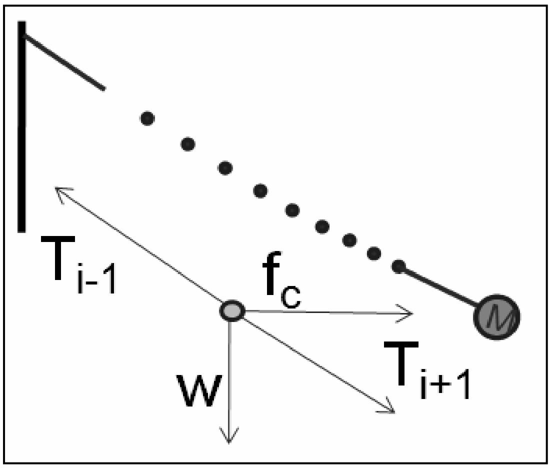

Figure 3: Discretization Assumption: to each point four forces are exerted.

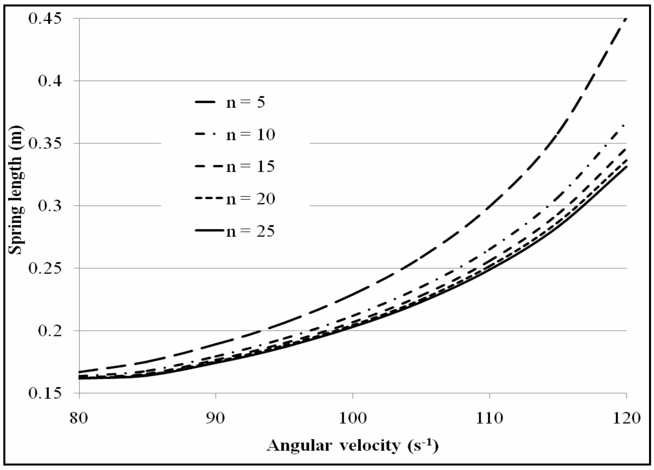

Figure 2: Mesh Independency Check

The numerical solution was developed in QBasic programming language. It takes about 10'000 iterations for each test case to reach convergence. To make sure about the independency of the results to the mesh size and the discretization number, this number was changed in a test case. The result (Figure 3) shows that the program's result approaches to a constant value when *n* increases. *n* = 20 was used in the rest of the test cases.

Physical Experiments

The relation between the spring length and angular velocity was investigated in physical experiments in several cases, to be compared with the numerical and analytical results.

The spring was rotated using a DC 12V motor, covering the spin range 100 to 400 RPM. The spin was variable by changing the input voltage to the motor. The connection between the motor and the spring was designed so that the end of the spring would be precisely in the axis of rotation; by putting the spring end inside a hole in the middle of the axis instead of sticking it to the side. Otherwise, standing wave motions could have been observed specially in cases with high angular velocities. The standing waves would make the spring a fully curved shape, with different parts of the spring in different sides of the axis.

Figure 4: Connection of the Motor and the spring

The angular velocity of the motor was measured by means of a tachometer. To measure the length of the spring, long exposure time photos were captured from the rotation, so that the entire motion of the spring in one round would be visible. The length of the spring would be measured by scaling the picture. (Figure 5)

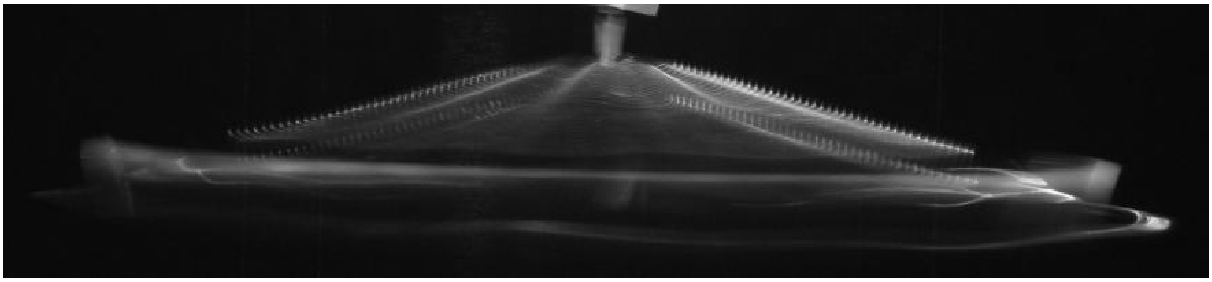

Figure 5: A Long Exposure Time Photo. The length of the spring was measured using similar pictures.

The modulus of the spring was measured by suspending masses with a spring of a known length, finding the spring constant and modulus. The mass of the spring was directly measured, used to find the linear density. These parameters were used as numerical input to the program and analytic solution to be compared with the experiments.

Discussion

The numerical theory showed perfect match with the physical experiments in several test cases. The length of the spring was measured while changing the angular velocity in one test case (Figure 6), in different initial lengths (Figure 7) and in different additional masses (figure 9).

The shape of the spring was also calculated numerically and compared with the experiments, (Figure 8). The shape of the spring is not fully linear; it has a slight curve upwards. The results all show an agreement between the numerical theory and the experiments.

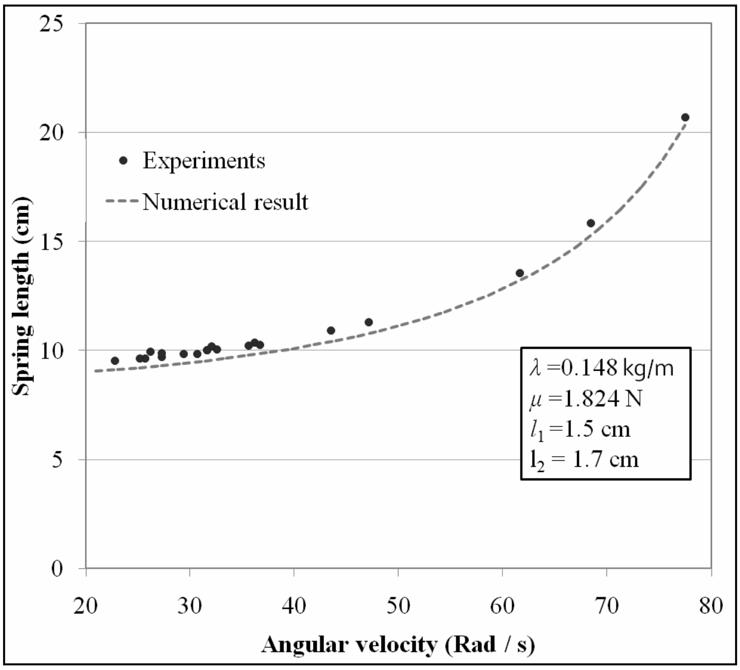

Figure 6: Comparison between the numerical results and physical experiments

Investigating different additional masses, the analytical theory was also compared with the experiments and the numerical method. As we see in (Figure 9), the result of the analytical solution does not match the physical experiments in small additional masses. However, in cases where the additional mass is greater than the mass of the spring, it could be assumed negligible and the analytical solution gives accurate answers.

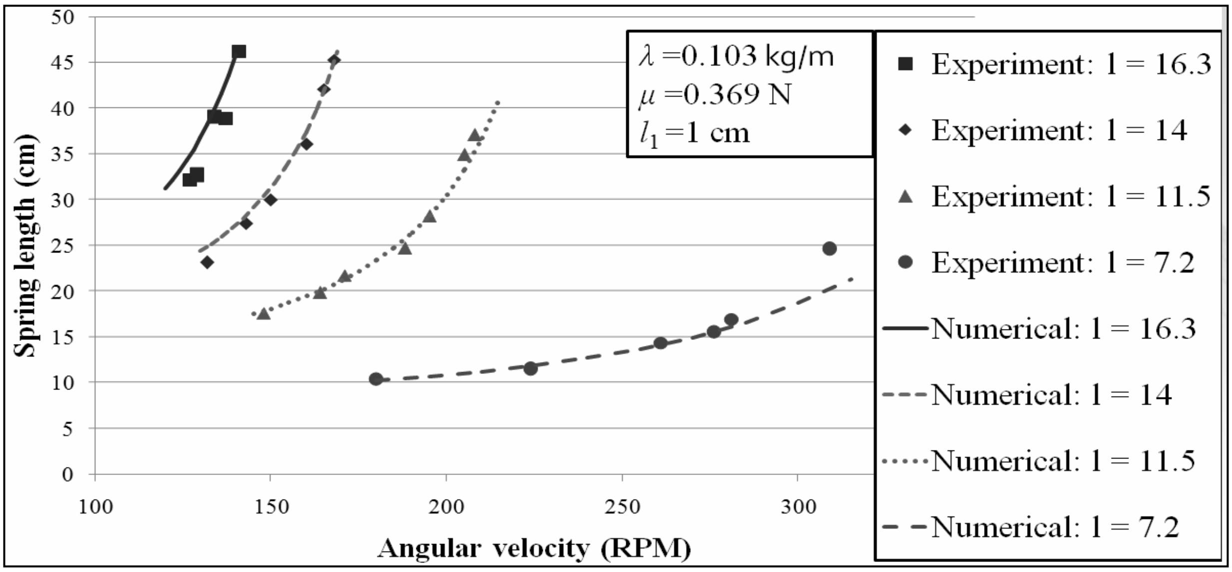

Figure 7: Comparing the numerical results and physical experiments in different initial lengths of the spring

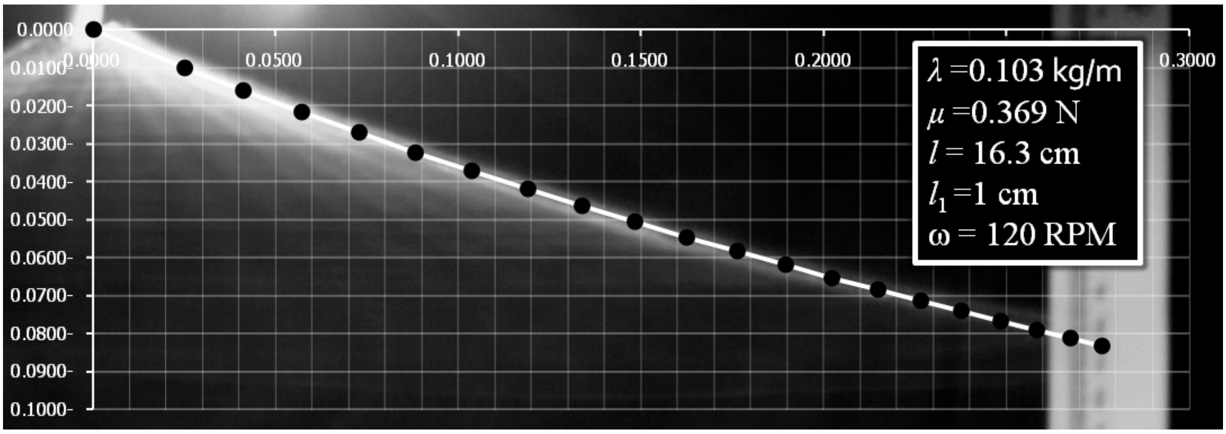

Figure 8: The results of the numerical theory and physical experiment on the shape of the spring

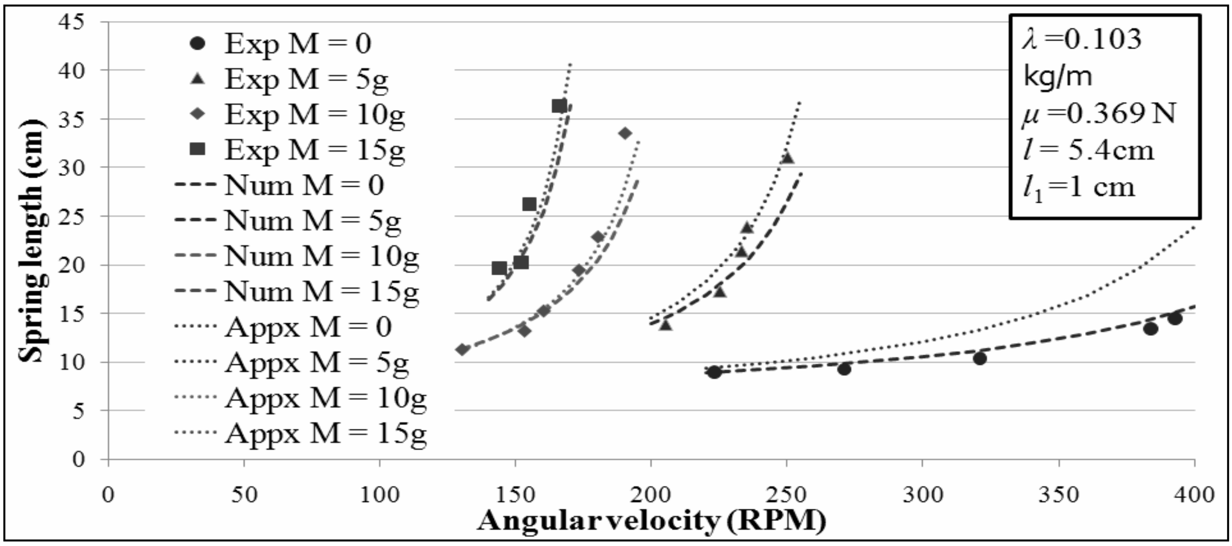

Figure 9: Comparing numerical and analytical theories with physical experiments in different additional masses

As a result, the method of the numerical approach used was fully proved experimentally in the experiment range; and the analytic solution was shown to be accurate in additional masses more than the spring mass.