第十三題：尖叫的棒子(Shrieking Rod)

A : The problem 13 of 2010 IYPT:

A metal rod is held between **two fingers** and hit. Investigate how the sound produced depends on **the position of holding and hitting** the rod?

There are two important points needed to be specified, one is the position of holding the rod and the other is the position of hitting the rod.

B : The wave equations of Longitudinal wave and Flexural wave

A rod supported at its center exhibits three type of vibrations : Longitudinal, Torsional, and Flexural, which depend on the bulk and shear properties of the material of the rod. Young's modulus for Aluminum is about 69 GPa, and the shear modulus is 26 GPa, while the density of Aluminum is 2700 kg/m^3. So, the velocity of longitudinal wave, c_L, is about 5055 m/s, and for the torsional wave is 3103 m/s. For the flexural wave, the frequency has more complicate form, as we will find later.

B-1: Longitudinal wave equation :

Consider the Fig.1 below.

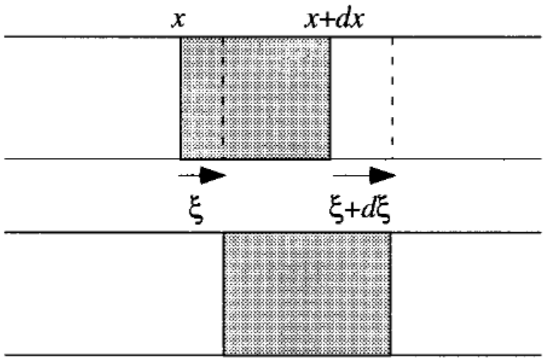

Fig. 1. Longitudinal strain in an infinitesimal stretched element of the rod.

An element of the rod lying between x and $x+dx$ is stretched so that its left edge displaces a distance ξ and its right edge displaces a distance $\xi+d\xi$. The *strain* is the relative change in length c, and the *stress* is the force per unit cross-sectional area F/A.

According to Hooke's law, stress is proportional to strain, so

$$\frac{F}{A} = -E\frac{\partial \xi}{\partial x}, \qquad \text{(Eq. 1)}$$

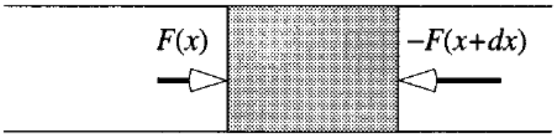

Fig. 2. Longitudinal stress. The force $-F(x+dx)$ on the right side of the element is the reaction to the force $+F(x+dx)$ on the rest of the rod to the right.

where E is the Young's elastic modulus. The negative sign ensures that, during compression, positive stress results in negative strain. The net force on the element is :

$$dF = F(x) - F(x+dx) = -\frac{\partial F}{\partial x}dx$$

According to the Newton's law,

$$dF = -\frac{\partial F}{\partial x}dx = dm\frac{\partial^2 \xi}{\partial t^2} \qquad \text{(Eq. 2)}$$

By combine the Hook's law of Eq. 1, we have:

$$F = -AE\frac{\partial \xi}{\partial x}, \text{ and } dF = AE\frac{\partial^2 \xi}{\partial x^2}dx,$$

due to the equation 2, the negative sign is cancelled. therefore we have :

$$dF = AE\frac{\partial^2 \xi}{\partial x^2}dx = dm\frac{\partial^2 \xi}{\partial t^2} = A\rho dx\frac{\partial^2 \xi}{\partial t^2}$$

where ρ is the density of rod. The second term and the forth term of the above equation can give the wave equation for longitudinal wave, and it is :

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c_L^2}\frac{\partial^2 \xi}{\partial t^2}, \qquad \text{(Eq. 3)}$$

where c_L is the longitudinal wave velocity, and equals to $c_L = \sqrt{\frac{E}{\rho}}$. Notice that both the group velocity and phase velocity of longitudinal waves are equal to $c_L = \frac{\omega}{k}$.

The solution of equation 2 is : $\xi(x,t) = \xi_0 \cos(kx - \omega t + \varphi)$, or

$$\xi(x,t) = \xi_0(\cos kx + \sin kx)\cos(\omega t + \varphi), \qquad \text{(Eq. 4)}$$

φ is the phase angle, and $c_L = \frac{\omega}{k}$.

When two fingers are holding at the center, then both ends of the rod are free. This implies that now stress is apply, then the stress at each end must vanish. So,

$0 = F = -AE\dfrac{\partial \xi}{\partial x}$, and this implies that : $\dfrac{\partial \xi}{\partial x} = 0$ at x = 0 and L. Apply the boundary conditions to the solution, (Eq. 4), we have

$$0 = \left.\frac{\partial \xi(x,t)}{\partial x}\right|_{x=0,L} = -\xi_0 k \sin(kx - \omega t + \varphi)\Big|_{x=0,L},$$

and it implies that $\sin kL = 0$. Thus the frequencies of the normal modes satisfy the condition $k_n L = n\pi$, thus:

$$f_n = n\frac{c_L}{2L} \qquad n = 1, 2, 3, \cdots \qquad \text{(Eq. 4)}$$

Eq. 4 represents the resonance condition of the longitudinal wave when the both ends of rod are free. So, when the two fingers hold the middle section of rod, some of the resonance mode will be depressed, especially when the holding position is on the antinode. As shown in the Fig. 3, the resonance longitudinal modes of n=1 to 5 are plotted.

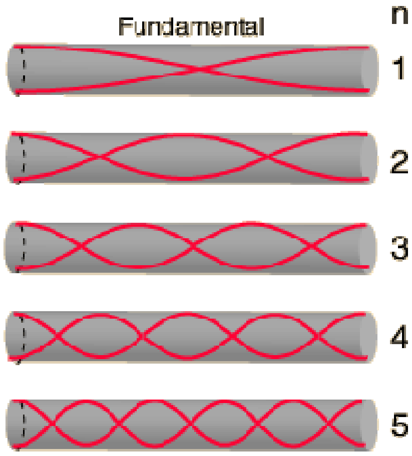

Fig. 3. Longitudinal modes of n=1 to 5 of a rod with two free ends.

For an one-meter-long 0.6-cm-diameter aluminum rod (E = 69 GPa, ρ = 2700 kg/m^3) with the two fingers holding the center position, the c_L is about 5055 m/s, and frequencies found are n=1, 2, 3, 4, 5, 6, as shown in Fig. 4. When two fingers hold the center of rod, the n=1, 3, 5 can be clearly found, however, the n= 2, 4 and 6 can also be found but with less intensity.

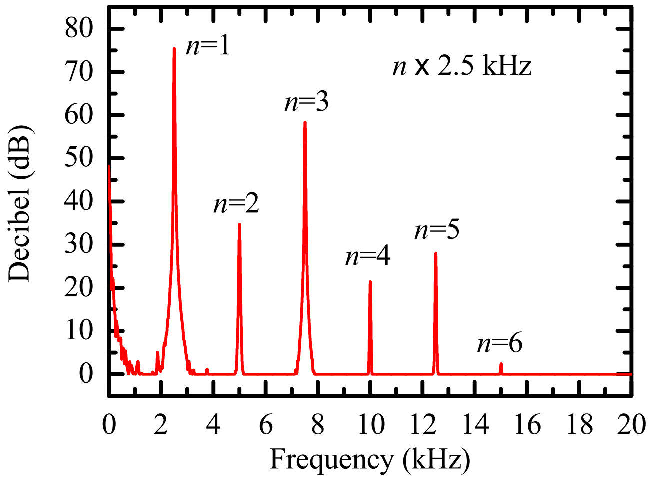

Fig. 4. Experimental result of 1-meter rod with two fingers holding at the center. The longitudinal frequencies observed are 2.5 kHz, to 15 kHz.

We can also use the two fingers to hold one end of rod. In this case, the end holding by two fingers is not totally free. The resonance longitudinal waves are shown in Fig. 5.

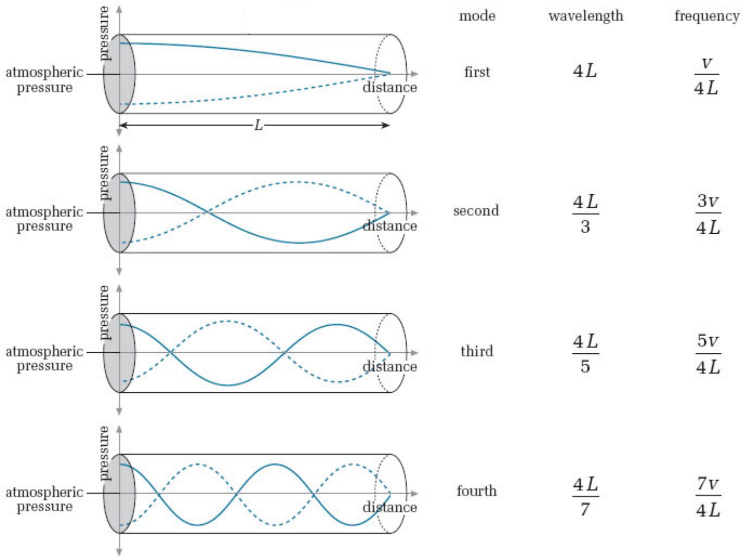

Fig. 5. Holding one end of the rod, and the resonance modes are found to have node at one end.

(We need some data to prove these modes can be detected. If these modes cannot be detected, then what is the reason??)

Similar to the derivation of longitudinal wave, the wave equation of tortional wave (same as shear wave) can be written as :

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c_T^2}\frac{\partial^2 \xi}{\partial t^2}, \qquad \text{(Eq.5)}$$

where c_T is the longitudinal wave velocity, and equals to $c_T = \sqrt{\frac{S}{\rho}}$. S is the shear modulus, and, for Aluminum, it is 26 GPa.

B-2: Flexural Wave Equation

When a rod is hit at one end, it is prone to vibrate transversely as well as longitudinally. Even a slight eccentricity in the collision or hitting of the rod results in significant flexural vibration. Moreover, flexural vibrations couple more efficiently to sound waves than longitudinal vibrations, because during flexural vibrations a larger surface area of the rod moves perpendicular to itself. We will assume that the rod is long and thin (so that the flexural wavelengths are large compared to the rod's diameter) and that the angles and displacements of the bends are small.

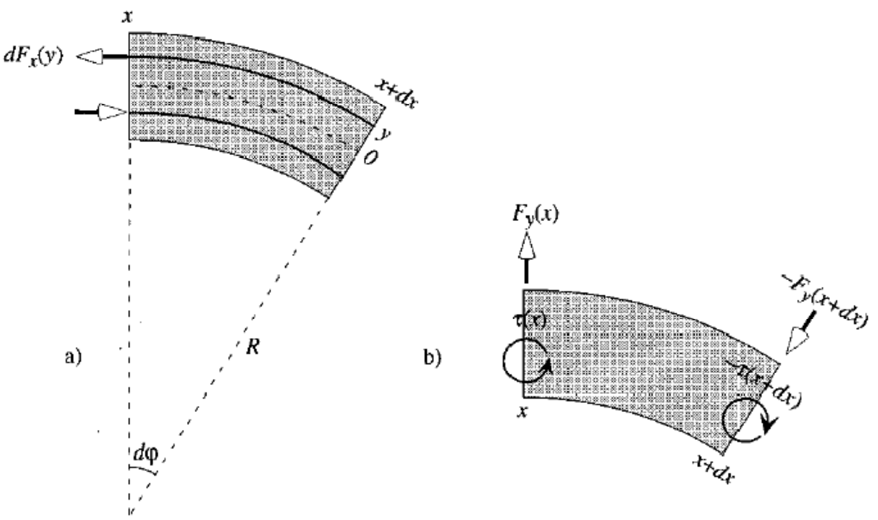

Fig. 6. Flexural stress and strain: (a) the strain in an infinitesimal bent element of the rod, and (b) the stress. The force $-F_y(x+dx)$ and torque $-\tau(x+dx)$ on the right side of the element is the reaction to the force $+F_y(x+dx)$ and torque $+\tau(x+dx)$ on the rest of the rod to the right.

As shown in Fig. 6, the outer part of the bent rod is stretched while the inner part is compressed. Assume that the point (x,y) displaces through (ξ,η). The longitudinal strain in an infinitesimal element a distance y from the neutral center line is $\partial\xi/\partial x$, and the longitudinal stress is dF_x/dA. According to Hooke's law, we have

$$\frac{dF_x}{dA} = -E\frac{\partial \xi}{\partial x} \qquad \text{(Eq. 6)}$$

dF_x is negative for outer filaments under tension and positive for inner filaments under compression. The strain can be expressed in terms of the radius of curvature R of the bend,

$$\frac{\partial \xi}{\partial x} = \frac{(R+y)d\varphi - Rd\varphi}{Rd\varphi} = \frac{y}{R} \qquad \text{(Eq. 7)}$$

Although the total longitudinal force integrated across the cross section vanishes (tension above canceling compression below), there is a nonvanishing moment of force or torque about the neutral axis, i.e. $\tau = \int d\tau = \int ydF_x$. So:

$$\tau = \int y(-E)\frac{y}{R}dA = \left[-\frac{EA}{R}\right] \times \left(\frac{1}{A}\int y^2 dA\right) = -\frac{EA}{R}R_G^2 \qquad \text{(Eq. 8)}$$

where R_G is the radius of gyration of the cross section. For a circular rod of diameter D, $R_G = D/4$. As for a square rod of side D, radius of gyration is $R_G = D/\sqrt{12}$.
Note: This is the Euler-Bernoulli results.
If the bending angles of the rod are small (as shown in Fig. 6), so that $\partial\eta/\partial x << 1$, we may approximate the curvature of the bend as $\frac{1}{R} \approx \frac{\partial^2 \eta}{\partial x^2}$, so that

$$\tau = -EAR_G^2 \frac{\partial^2 \eta}{\partial x^2} \qquad \text{(Eq. 9)}$$

By computing the torques about the left end of the element, we require the balance

$$F_y(x+dx)dx = \tau(x) - \tau(x+dx), \text{ or } F_y = -\frac{\partial \tau}{\partial x} = EAR_G^2 \frac{\partial^3 \eta}{\partial x^3} \qquad \text{(Eq.10)}$$

The net transverse force on the element is

$dF_y = F_y(x) - F_y(x+dx) = -\frac{\partial F_y}{\partial x}dx = dm\frac{\partial^2 \eta}{\partial t^2} = \rho A dx \frac{\partial^2 \eta}{\partial t^2}$, and to simplify the formula, we have :

$-AER_G^2 dx\frac{\partial^4 \eta}{\partial x^4} = \rho A dx \frac{\partial^2 \eta}{\partial x^2}$, and this gives $\frac{\partial^4 \eta}{\partial x^4} = -\frac{\rho}{ER_G^2}\frac{\partial^2 \eta}{\partial t^2} = -\frac{1}{c_L^2 R_G^2}\frac{\partial^2 \eta}{\partial t^2}$, so:

$$\frac{\partial^4 \eta}{\partial x^4} = -\frac{1}{c_L^2 R_G^2}\frac{\partial^2 \eta}{\partial t^2} \qquad \text{(Eq.10)}$$

Equation 10 is also called Euler-Bernoulli equation for flexural vibrations, with the

$R_G = D/4$, and $R_G = D/\sqrt{12}$ for a circular rod with diameter D and a square rod of side D, respectively.

When the both ends of rod are free, and there is no shearing or applied bending moment force at the free end, then the torque and shear force must vanish at each end, so : $0 = \tau = -AER_G^2 \frac{\partial^2 \eta}{\partial x^2}$, and $0 = F_y = AER_G^2 \frac{\partial^3 \eta}{\partial x^3}$ at x = 0, and L. The solution of Eq. 10 is :$\eta(x,t) = [\alpha \cos(kx) + \beta \sinh(kx) + \gamma \cos(kx) + \delta \sin(kx)]\cos(\omega t + \varphi)$, and the dispersion relation is : $\omega = k^2 c_L R_G$, The quadratic relation between the temporal and spatial frequencies indicates the flexural wave dispersive. In a nondispersive wave medium, waves can propagate without deformation, however, for the dispersive wave, the wave deforms as function of time and position. A discussion can be found in the following web site :

http://physics.usask.ca/~hirose/ep225/animation/dispersion/anim-dispersion.html

The boundary condition for the even symmetric mode gives $\tan(kL/2) = -\tanh(kL/2)$, and for the odd modes, $\tan(kL/2) = +\tanh(kL/2)$. The k values satisfying both even and odd modes are : $k = m\pi/(2L)$, m = 3, 7, 11,… are for the odd modes and m = 5, 9, 13,… are for the even modes. As shown in the Fig. 7, the intersections of $\tan(kL/2) = +\tanh(kL/2)$, are found for m = 3, 7, 11,…, and $\tan(kL/2) = -\tanh(kL/2)$, are found for m = 5, 9, 13,….

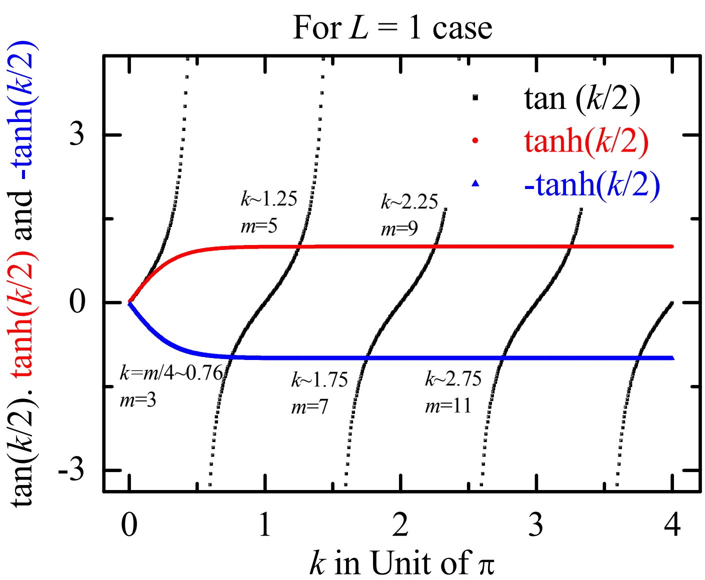

Fig. 6 : The solution of the even symmetric mode, $\tan(kL/2) = -\tanh(kL/2)$, and

the odd modes, $\tan(kL/2) = +\tanh(kL/2)$. The L value is set to be 1, for simplicity.

So the predicted frequencies are $f_m = \frac{1}{2\pi}\left(\frac{m\pi}{2L}\right)^2 c_L R_G = m^2 \frac{\pi c_L R_G}{8L^2}$, when both ends are free.

NOTE: For circular rod the $R_G = D/4$, and $R_G = D/\sqrt{12}$ for the square rod.

<u>If one end of rod is clamped and the other end is free</u>, then :

$$\eta = \frac{\partial \eta}{\partial x} = 0 \quad at \quad x = 0, \quad \text{and} \quad \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^3 \eta}{\partial x^3} = 0 \quad at \quad x = L,$$

and then : $f_n = n^2 \cdot \frac{\pi c_L R_G}{8L^2}$, and n =1.19366, 2.98829, 5.00001, 7, 9, 11…. , and they are : $f_1 = (1.19366)^2 \cdot \frac{\pi c_L R_G}{8L^2}$

$f_2 = (2.98829)^2 \cdot \frac{\pi c_L R_G}{8L^2}$

$f_3 = (5.00001)^2 \cdot \frac{\pi c_L R_G}{8L^2}$

For $f_{n>3}$, n is approximately equal to the odd number!

Appendix A : Solve the Differential Equation of the Flexural Wave

The flexural wave equation can be written as :

$$\frac{\partial^4 \eta}{\partial x^4} = -\frac{1}{c_L^2 R_G^2}\frac{\partial^2 \eta}{\partial t^2} \qquad \text{(A-1)}$$

Since the Young's Modulus for most metal is several tenth of GPa, therefore the simple harmonics application can be applied. By using a general solution of the equation (A-1), that is $\sim \exp(kx)\exp(-i\omega t)$, then (A-1) can be expressed as :

$$k^4 = \frac{1}{c_L^2 R_G^2}\omega^2 \qquad \text{(A-2)}$$

Certainly, this equation implies the dispersive relation $\omega = k^2 c_L R_G$, as we explained in the text. There are four roots for the *k* in equation (A-2), and they are :

$\sqrt[4]{\frac{\omega^2}{c_L^2 R_G^2}}$, $-1\times\sqrt[4]{\frac{\omega^2}{c_L^2 R_G^2}}$, $i\times\sqrt[4]{\frac{\omega^2}{c_L^2 R_G^2}}$, and $-i\times\sqrt[4]{\frac{\omega^2}{c_L^2 R_G^2}}$.

The general solution of (A-1) is : (if $k = \sqrt[4]{\frac{\omega^2}{c_L^2 R_G^2}}$)

$$\eta = C_1\exp(kx) + C_2\exp(-kx) + C_3\exp(ikx) + C_4\exp(-ikx)$$

Or :

$$\eta(x,t) = [\alpha\cos(kx) + \beta\sinh(kx) + \gamma\cos(kx) + \delta\sin(kx)]\cos(\omega t + \varphi) \qquad \text{(A-3)}$$

A: For the fixed end condition : $\eta = \frac{\partial \eta}{\partial x} = 0 \quad at \quad x = 0$, we can get :

$\delta = -\beta$ and $\gamma = -\alpha$.

B: For the free end condition : $\frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^3 \eta}{\partial x^3} = 0 \quad at \quad x = 0$, we can get :

$\delta = \beta$ and $\gamma = \alpha$.

C: For both ends of rod are free, we will have :

$$\alpha(\cosh kL - \cos kL) = \beta(\sin kL - \sinh kL), \text{ and}$$

$$\alpha(\sinh kL + \sin kL) = \beta(\cos kL - \cosh kL).$$

These two equations give : $\tan(kL/2) = \pm\tanh(kL/2)$ to give allowed *k* values.

D: For one end is fixed and one end is free, we will have $\cosh kL \times \cos kL = -1$

E: For both end are fixed, we will have $\frac{\sinh kL - \sin kL}{\cos kL - \cosh kL} = \frac{\cos kL - \cosh kL}{\sinh kL + \sin kL}$. This is the equation that give allowed values of k.